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LETTER TO THE EDITOR

Addition of arbitrary number of identical angular momenta

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Abstract. An alternate method is used to establish the analytic expression recently given and incompletely proved by Mikhailov for the multiplicity of occurrence of any angular momentum in the vector addition of n identical angular momenta.

1. Introduction

Mikhailov (1977) has recently given an analytic expression for the multiplicity P^s_{jn} of occurrence of any angular momentum j in the decomposition of the direct product of n identical angular momenta s. He has guessed the expression by examining special cases for low values of n. However, a proof based on combinatorics has been provided only for $j \geq s$. In the following, we shall present another method of arriving at this expression which does not distinguish between the two cases $j \geq s$ and $j < s$. In order to enable easy comparison with Mikhailov's work, we have used the notation of his paper throughout.

2. Proof of the expression for the multiplicity

The irreducible representations of the rotation group in three dimensions are characterised by a number s which takes the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The irreducible representation characterised by the number s is $(2s+1)$-dimensional and its character $\chi^{(s)}(\theta)$ is given by

$$\chi^{(s)}(\theta) = \frac{\sin(s+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \tag{1}$$

(Hamermesh 1962, equation (9.41)) where θ is the angle of rotation. (All rotations through the same angle θ about different axes through the same centre have the same character. From group theory point of view, they belong to the same class.) The characters of different irreducible representations (called the irreducible characters) are orthogonal over the range $(0, \pi)$ with the weight function $(1-\cos\theta)$. Indeed

$$\frac{1}{\pi}\int_0^{\pi} \chi^{(s)}(\theta)\chi^{(s')}(\theta)(1-\cos\theta)\,\mathrm{d}\theta = \delta_{ss'} \tag{2}$$

(Hamermesh 1962, equation (9.28)).

Now the character of the direct product of n identical angular momenta s is

$$(\chi^{(s)}(\theta))^n = \left(\frac{\sin (s+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}\right)^n \tag{3}$$

and the multiplicity P^s_{jn} which we wish to determine appears in the decomposition

$$(\chi^{(s)}(\theta))^n = \sum_{j=j_0}^{ns} P^s_{jn}\chi^{(j)}(\theta) \tag{4}$$

where $j_0 = 0$ or $\frac{1}{2}$ depending upon whether ns is integer or half integer and j moves in steps of 1 from j_0 to ns.

The orthogonality (equation (2)) of the irreducible characters then results in an integral representation for the multiplicity, namely

$$P^s_{jn} = \frac{1}{\pi}\int_0^{\pi} \left(\frac{\sin (s+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}\right)^n \frac{\sin (j+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}(1-\cos\theta)\,d\theta. \tag{5}$$

Since the integrand in the above expression is an even function of θ, we can rewrite

$$P^s_{jn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\sin (s+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}\right)^n \frac{\sin (j+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}(1-\cos\theta)\,d\theta. \tag{6}$$

Noting that $\int_{-\pi}^{\pi} O(\theta)E(\theta)\,d\theta = 0$ where

$$O(-\theta) = -O(\theta), \qquad E(-\theta) = E(\theta),$$

and replacing $1-\cos\theta$ by $2\sin^2\frac{1}{2}\theta$, we simplify the above expression to

$$P^s_{jn} = \frac{1}{\pi \mathrm{i}}\int_{-\pi}^{\pi} \frac{[\sin (s+\frac{1}{2})\theta]^n}{(\sin\frac{1}{2}\theta)^{n-1}} \exp[\mathrm{i}(j+\tfrac{1}{2})\theta]\,d\theta. \tag{7}$$

To evaluate the above integral, we transform it into a contour integral by writing $e^{i\theta} = Z$, which results in

$$P^s_{jn} = \frac{1}{2\pi \mathrm{i}}\oint (Z^{-(s+\frac{1}{2})} - Z^{(s+\frac{1}{2})})^n (Z^{-1/2} - Z^{1/2})^{-(n-1)} Z^{j-\frac{1}{2}}\,dZ \tag{8}$$

where the contour is the circle $|Z| = 1$. Expanding the integrand (which is analytic except for a pole at $Z = 0$, of order $ns - j + 1$) in powers of Z, P^s_{jn} becomes (for $n \geq 2$)†

$$P^s_{jn} = \frac{1}{2\pi \mathrm{i}}\oint \sum_{k,l\geq 0} {}^nC_k\, {}^{-(n-1)}C_l (-1)^{k+l} Z^{-(n-2k)(s+\frac{1}{2})+l+\frac{1}{2}n+j-1} \tag{9}$$

Performing the integration term by term, we note that we require that k and l *must* satisfy (*in addition to being non-negative*)

$$-(n-2k)(s+\tfrac{1}{2}) + l + \tfrac{1}{2}n + j = 0$$

to provide for a non-zero contribution. Thus we arrive at

$$P^s_{jn} = \sum_{0\leq k\leq[(ns-j)/(2s+1)]} (-1)^k\, {}^nC_k C^{n(s+1)-k(2s+1)-j-2}_{n-2} \tag{10}$$

where $[(ns-j)/(2s+1)]$ is the greatest integer less than or equal to $(ns-j)/(2s+1)$ and

† Trivially $P^s_{j0} = \delta_{j0}$ and $P^s_{j1} = \delta_{sj}$. The formula (10) can indeed be interpreted for these cases also although it is not necessary.

we have used ${}^{-(n-1)}C_l = (-1)^l C_{n-2}^{n+l-2}$. The above is the same expression for P_{jn}^s as obtained by Mikhailov (1977).

Incidentally equation (24) in Mikhailov (1977), which he could not prove, represents the obvious result

$$\int_{-\pi}^{\pi} \sin (j-s)\theta \left(\frac{\sin (s+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}\right)^{n-1} \mathrm{d}\theta = 0$$

when we try to evaluate it using the contour integration technique as employed in the present paper.

References

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